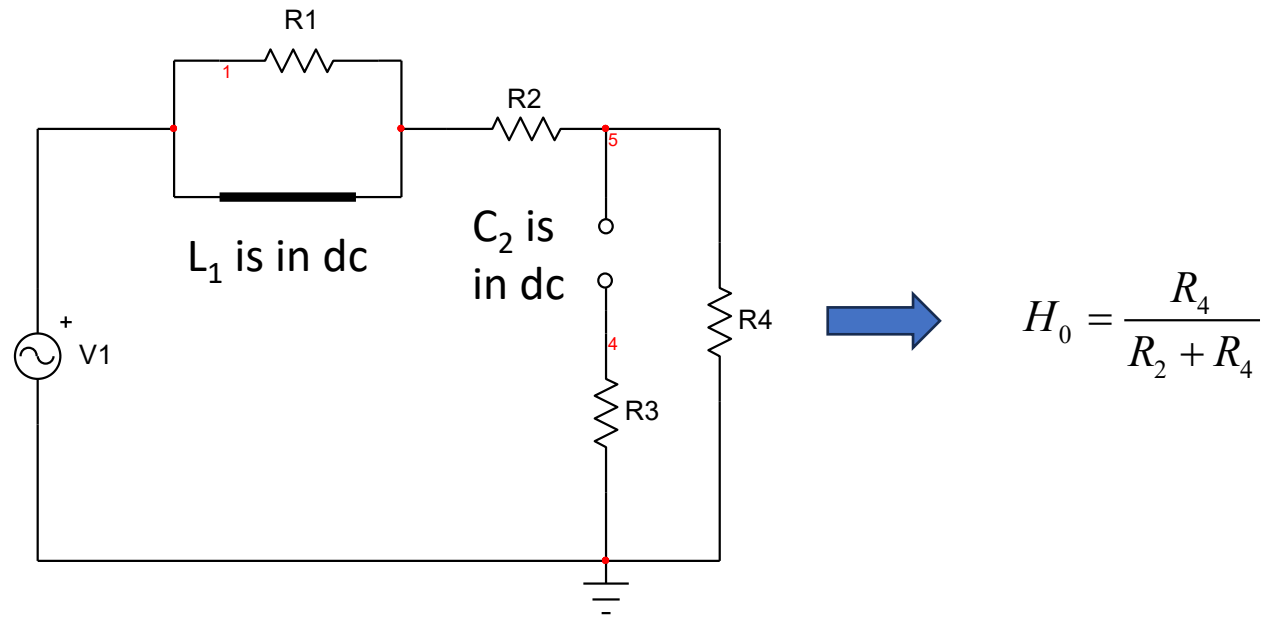
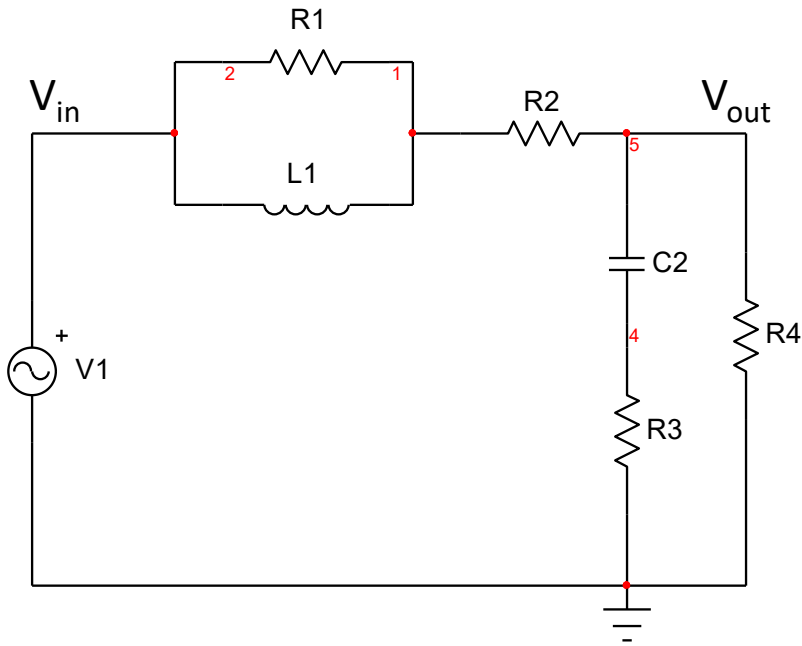


In these examples, I am showing how to solve a circuit with two different reference states in a 2<sup>nd</sup>-order circuit.

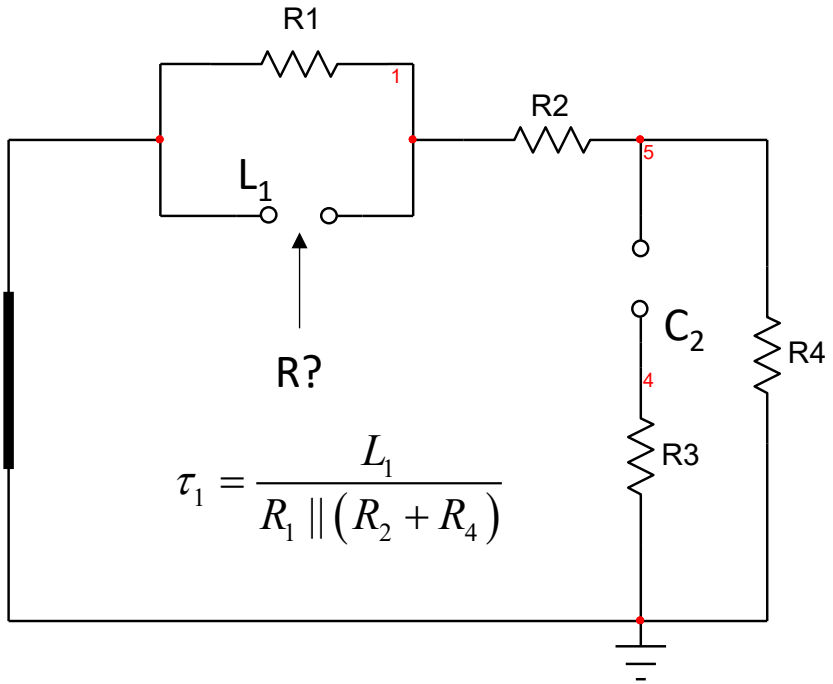
We start by considering  $s = 0$  for the reference state:



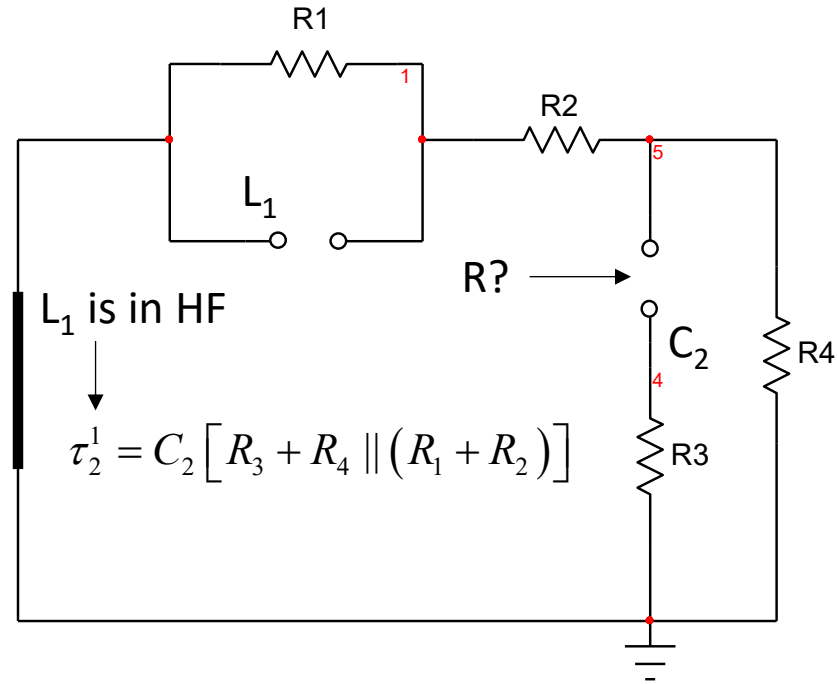
$$H_0 = \frac{R_4}{R_2 + R_4}$$

Brute-force TF

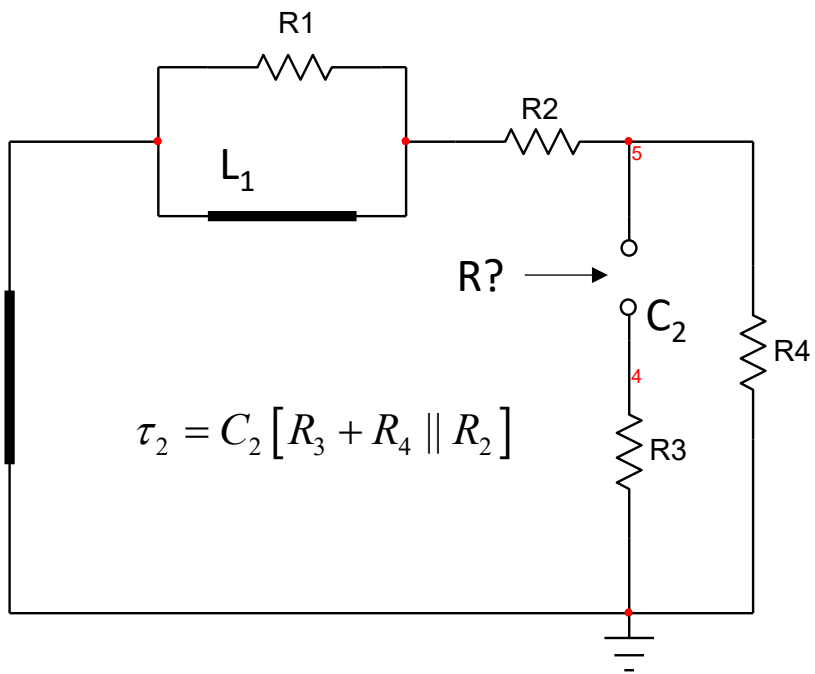
$$H_{BF}(s) := \frac{\left[ R_4 \parallel \left( R_3 + \frac{1}{s \cdot C_2} \right) \right]}{\left[ R_4 \parallel \left( R_3 + \frac{1}{s \cdot C_2} \right) \right] + \left[ R_2 + (s \cdot L_1) \parallel R_1 \right]}$$



In this case,  $C_2$  is in its reference state.

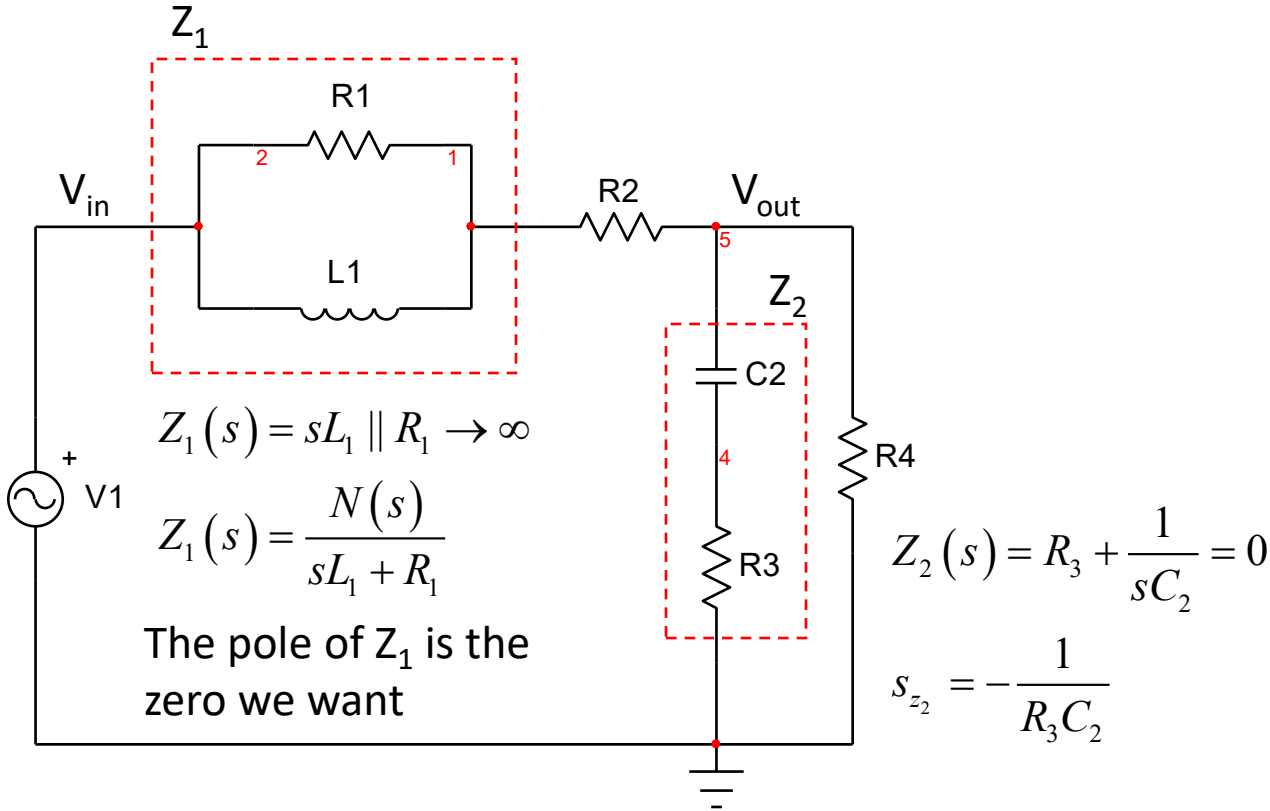


In this mode,  $L_1$  is in the opposite of its reference state (which is high-frequency).



In this case,  $L_1$  is in its reference state.

For the zeroes, there are two and can be obtained by inspection immediately:



$$\omega_{z_1} = \frac{R_1}{L_1} \quad \omega_{z_2} = \frac{1}{R_3C_2}$$

$$N(s) = \left(1 + \frac{s}{\omega_{z_1}}\right) \left(1 + \frac{s}{\omega_{z_2}}\right)$$

$$D(s) = 1 + b_1s + b_2s^2$$

$$b_1 = \tau_1 + \tau_2 = \frac{L_1}{R_1 \parallel (R_2 + R_4)} + C_2 (R_3 + R_4 \parallel R_2)$$

$$b_2 = \tau_1\tau_2 = \frac{L_1}{R_1 \parallel (R_2 + R_4)} C_2 [R_3 + R_4 \parallel (R_1 + R_2)]$$

$$H(s) = H_0 \frac{\left(1 + \frac{s}{\omega_{z_1}}\right) \left(1 + \frac{s}{\omega_{z_2}}\right)}{1 + \frac{s}{\omega_0 Q} + \left(\frac{s}{\omega_0}\right)^2}$$

$$H_0 = \frac{R_4}{R_1 + R_2 + R_4}$$

$$\omega_{z_1} = \frac{R_1}{L_1} \quad \omega_{z_2} = \frac{1}{R_3C_2}$$

$$\|(x, y) := \frac{x \cdot y}{x + y}$$

$$R_1 := 10\text{k}\Omega \quad R_2 := 1\text{k}\Omega \quad R_3 := 10\Omega \quad R_4 := 500\Omega \quad C_2 := 0.1\mu\text{F} \quad L_1 := 100\mu\text{H}$$

$$\tau_1 := \frac{L_1}{R_1 \parallel (R_2 + R_4)} = 0.077 \cdot \mu\text{s}$$

$$H_0 := \frac{R_4}{R_2 + R_4}$$

$$\tau_2 := C_2 \cdot (R_3 + R_4 \parallel R_2) = 34.333 \cdot \mu\text{s}$$

$$\tau_{12} := C_2 \cdot [R_3 + R_4 \parallel (R_2 + R_1)] = 48.826 \cdot \mu\text{s}$$

$$\omega_{z1} := \frac{R_1}{L_1} \quad \omega_{z2} := \frac{1}{R_3 \cdot C_2}$$

$$b_1 := \tau_1 + \tau_2 \quad b_2 := \tau_1 \cdot \tau_{12}$$

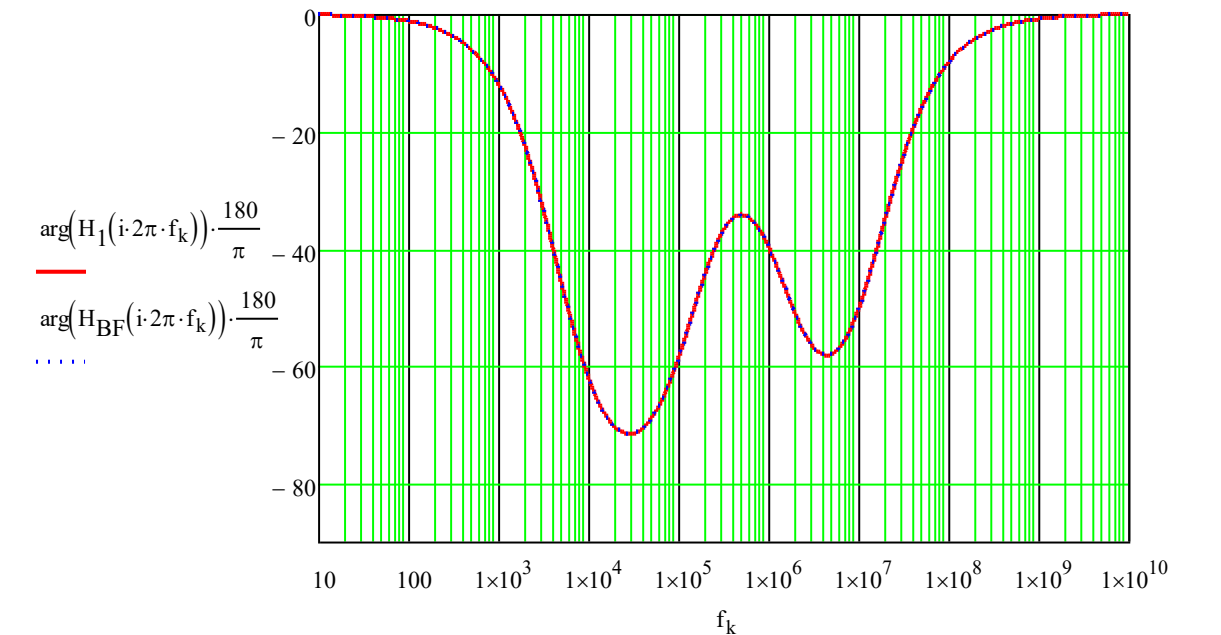
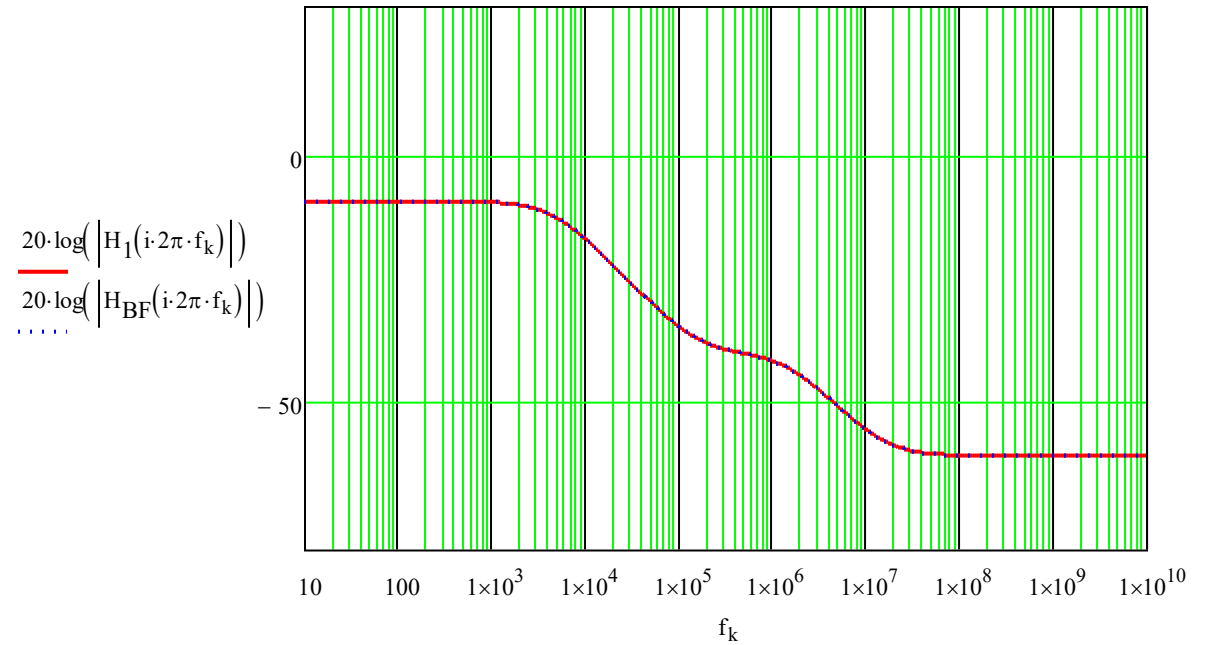
$$D_1(s) := 1 + s \cdot (\tau_1 + \tau_2) + s^2 \cdot (\tau_1 \cdot \tau_{12}) \quad N_1(s) := \left(1 + \frac{s}{\omega_{z1}}\right) \cdot \left(1 + \frac{s}{\omega_{z2}}\right)$$

$$Q := \frac{\sqrt{b_2}}{b_1} = 0.056 \quad \omega_0 := \frac{1}{\sqrt{b_2}} \quad f_0 := \frac{\omega_0}{2 \cdot \pi} = 82.26 \cdot \text{kHz}$$

$$D_2(s) := 1 + \frac{s}{\omega_0 \cdot Q} + \left(\frac{s}{\omega_0}\right)^2$$

$$H_1(s) := H_0 \cdot \frac{N_1(s)}{D_1(s)}$$

$$H_{\text{BF}}(s) := \frac{\left[ R_4 \parallel \left( R_3 + \frac{1}{s \cdot C_2} \right) \right]}{\left[ R_4 \parallel \left( R_3 + \frac{1}{s \cdot C_2} \right) \right] + \left[ R_2 + (s \cdot L_1) \parallel R_1 \right]}$$



The factorization of the 2<sup>nd</sup>-order polynomial form changes depending on the adopted leading term:

2<sup>nd</sup>-order expression

$$H(s) = \frac{a_0 + a_1s + a_2s^2}{b_0 + b_1s + b_2s^2}$$

↑   ↑   ↑  
dc mid-freq high-freq

$$H(s) = \frac{\frac{a_0}{b_0} \left( 1 + \frac{a_1}{a_0}s + \frac{a_2}{a_0}s^2 \right)}{1 + \frac{b_1}{b_0}s + \frac{b_2}{b_0}s^2}$$

↑  
dc

$$H(s) = \frac{\frac{a_1}{b_1} \left( \frac{a_0}{a_1} + s + \frac{a_2}{a_1}s^2 \right)}{b_0 + s + \frac{b_2}{b_1}s^2}$$

↑  
Mid-band

Ok for a bandpass filter

$$H(s) = \frac{\frac{a_2}{b_2} \left( \frac{a_0}{a_2} + \frac{a_1}{a_2}s + s^2 \right)}{\frac{b_0}{b_2} + \frac{b_1}{b_2}s + s^2}$$

↑  
HF



Factor  $s^2$

$$H_{11}(s) := \frac{R_4 + \left( \frac{L_1 \cdot R_4}{R_1} + C_2 \cdot R_3 \cdot R_4 \right) \cdot s + \frac{C_2 \cdot L_1 \cdot R_3 \cdot R_4 \cdot s^2}{R_1}}{R_2 + R_4 + s \cdot (R_2 + R_4) \cdot \left[ \frac{L_1}{R_1 \parallel (R_2 + R_4)} + C_2 \cdot (R_3 + R_4 \parallel R_2) \right] + s^2 \cdot (R_2 + R_4) \cdot \left[ \frac{L_1}{R_1 \parallel (R_2 + R_4)} \cdot [C_2 \cdot [R_3 + R_4 \parallel (R_2 + R_1)]] \right]}$$

$$\underline{a_0} := R_4 \quad a_1 := \frac{L_1 \cdot R_4}{R_1} + C_2 \cdot R_3 \cdot R_4 \quad a_2 := \frac{C_2 \cdot L_1 \cdot R_3 \cdot R_4}{R_1}$$

$$b_0 := R_2 + R_4 \quad \underline{b_1} := (R_2 + R_4) \cdot \left[ \frac{L_1}{R_1 \parallel (R_2 + R_4)} + C_2 \cdot (R_3 + R_4 \parallel R_2) \right] \quad \underline{b_2} := (R_2 + R_4) \cdot \left[ \frac{L_1}{R_1 \parallel (R_2 + R_4)} \cdot [C_2 \cdot [R_3 + R_4 \parallel (R_2 + R_1)]] \right]$$

$$H_{12}(s) := \frac{a_0 + a_1 \cdot s + a_2 \cdot s^2}{b_0 + b_1 \cdot s + b_2 \cdot s^2}$$

$$H_{00} := 20 \cdot \log \left( \frac{a_0}{b_0} \right) = -9.542$$

dc gain

$$H_{MB} := 20 \cdot \log \left( \frac{a_1}{b_1} \right) = -40.19$$

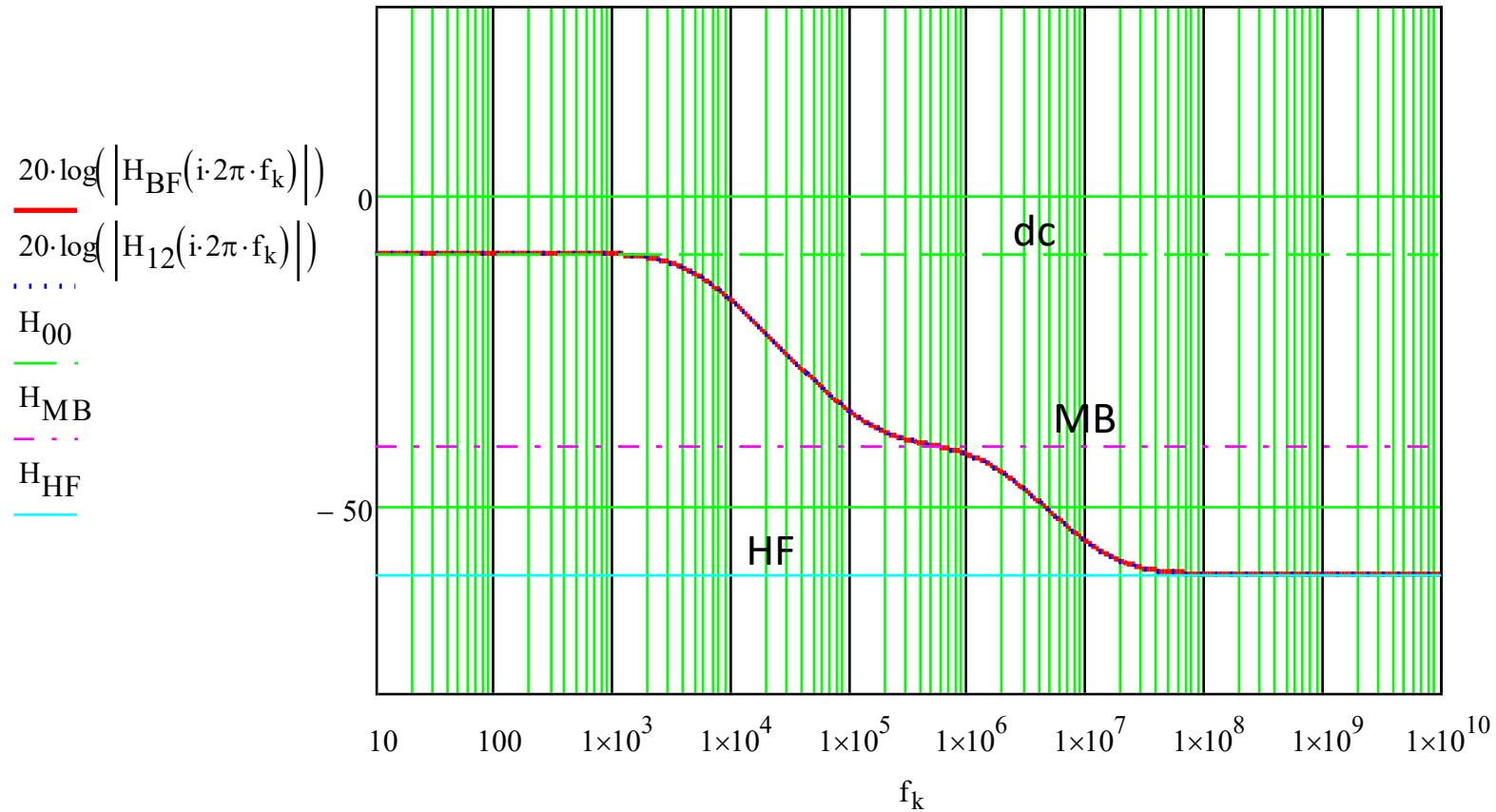
mid-band gain

$$H_{HF} := 20 \cdot \log \left( \frac{a_2}{b_2} \right) = -61.008$$

high-frequency gain

$$H(s) = \frac{a_2}{b_2} \frac{s^2 \frac{a_0}{a_2 s^2} + \frac{a_1}{a_2 s^2} s + 1}{\frac{b_0}{b_2 s^2} + \frac{b_1}{b_2 s^2} s + 1}$$

$$H(s) = \frac{a_2}{b_2} \frac{1 + \frac{a_1}{a_2 s} + \frac{a_0}{a_2 s^2}}{1 + \frac{b_1}{b_2 s} + \frac{b_0}{b_2 s^2}}$$



Depending on the adopted leading term, it is possible to highlight any of the three possible gains: the dc gain when  $s = 0$ , the midband gain for moderate frequencies and the high-frequency gain when  $s$  approaches infinity.

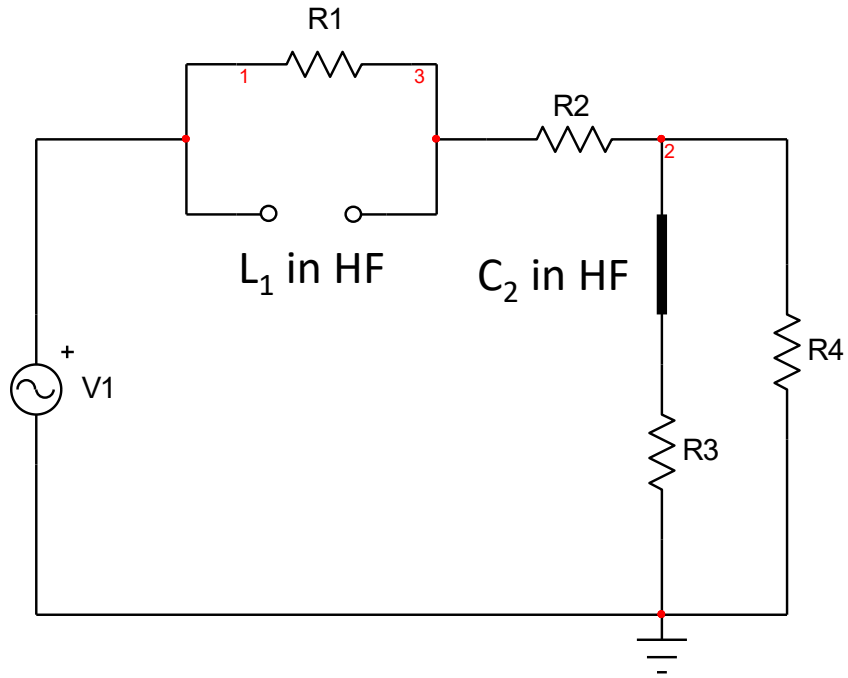
If the dc gain is nonzero, we can use the classical approach with  $H_0$  as a leading term. If the dc gain is zero (or not) but there exists a high-frequency gain, then use the adapted expression with  $H_{INF}$  as a leading term. And if there are no dc or HF gains, like in a selective filter or bandpass, use the expression with  $H_{MB}$  as the leading term.

Now we change the reference state to  $s$  approaches infinity:

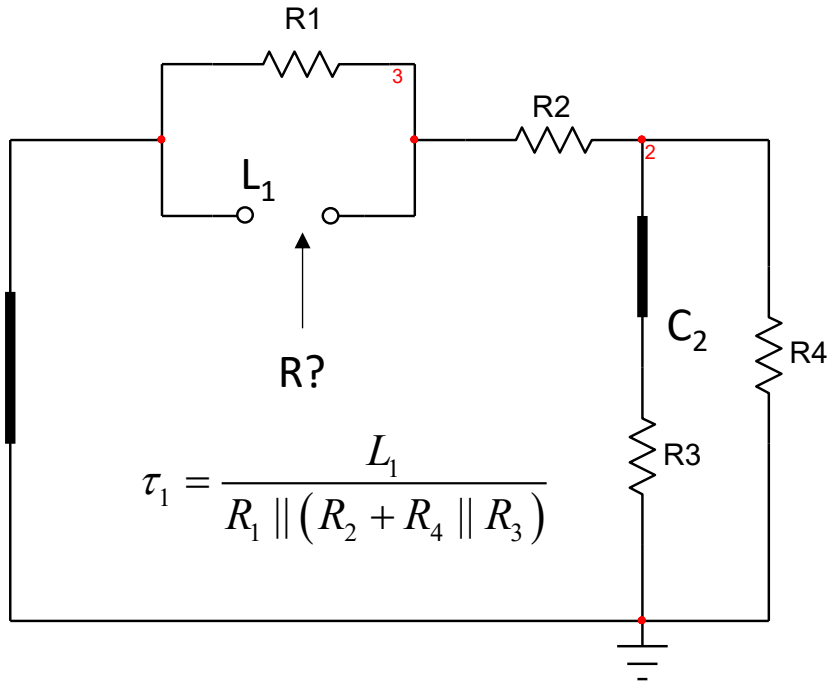
inductance  $L_1$  is open  
 $C_2$  is a short.



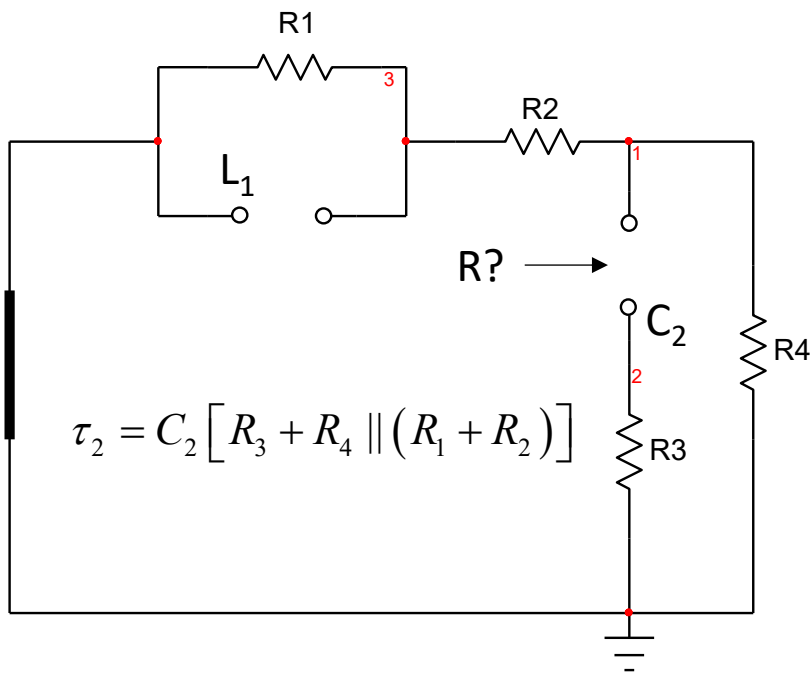
This is the new  
reference state



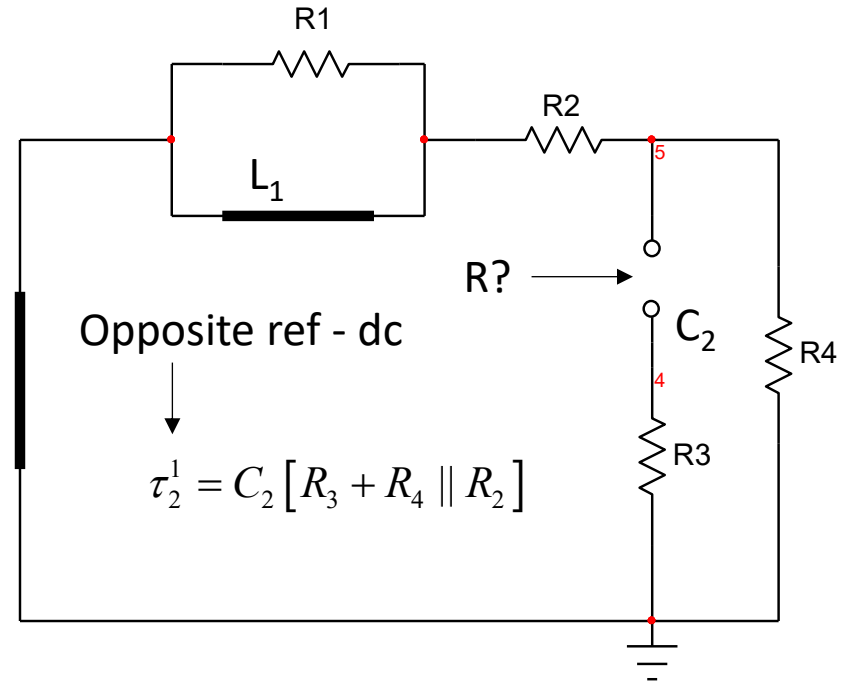
$$H_{INF} = \frac{R_3 \parallel R_4}{R_3 \parallel R_4 + R_1 + R_2}$$



In this case,  $C_2$  is in its reference state which is a short.



In this case,  $L_1$  is in its reference state which is open.



In this mode,  $L_1$  is in the opposite of its reference state (which is a short).

$$b_1 = \frac{1}{\tau_1} + \frac{1}{\tau_2} \quad b_2 = \frac{1}{\tau_1 \tau_2}$$

$$D(s) = 1 + \frac{b_1}{s} + \frac{b_2}{s^2} = 1 + s \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) + \frac{1}{\tau_1 \tau_2 s^2} = 1 + \frac{\omega_0}{sQ} + \left( \frac{\omega_0}{s} \right)^2$$

$$Q = \frac{\sqrt{b_2}}{b_1} \quad \omega_0 = \sqrt{b_2}$$

The definitions for  $b_i$  and  $a_i$  supersede the ones given in the starting 2<sup>nd</sup>-order expression.

The zeroes are the same than the ones found by inspection but they factor differently to account for the new format. They are identical in position but the numerator is now factored with inverted zeroes:

$$N(s) = \left( 1 + \frac{\omega_{z_1}}{s} \right) \left( 1 + \frac{\omega_{z_2}}{s} \right)$$

The final 2<sup>nd</sup>-order transfer function with a high-frequency leading term is expressed as:

$$H(s) = H_{INF} \frac{\left( 1 + \frac{\omega_{z_1}}{s} \right) \left( 1 + \frac{\omega_{z_2}}{s} \right)}{1 + \frac{\omega_0}{sQ} + \left( \frac{\omega_0}{s} \right)^2}$$

In this expression,  $Q$  and  $\omega_0$  are those determined above and differ from the ones we have found earlier.

$$H_{\text{inf}} := \frac{R_3 \parallel R_4}{R_3 \parallel R_4 + R_1 + R_2} \quad 20 \cdot \log(H_{\text{inf}}) = -61.008 \quad \text{dB}$$

$$\tau_{1a} := \frac{L_1}{R_1 \parallel (R_2 + R_4 \parallel R_3)} = 0.109 \cdot \mu\text{s}$$

$$\tau_{2a} := C_2 \cdot [R_3 + R_4 \parallel (R_2 + R_1)] = 48.826 \cdot \mu\text{s}$$

$$\tau_{12a} := C_2 \cdot (R_3 + R_4 \parallel R_2) = 34.333 \cdot \mu\text{s}$$

$$D_{1a}(s) := 1 + \left( \frac{1}{\tau_{1a}} + \frac{1}{\tau_{2a}} \right) \cdot \frac{1}{s} + \left( \frac{1}{\tau_{1a} \cdot \tau_{12a}} \right) \cdot \frac{1}{s^2} \quad N_{1a}(s) := \left( 1 + \frac{\omega_{z1}}{s} \right) \cdot \left( 1 + \frac{\omega_{z2}}{s} \right)$$

$$b_{1a} := \frac{1}{\tau_{1a}} + \frac{1}{\tau_{2a}} \quad b_{2a} := \frac{1}{\tau_{1a} \cdot \tau_{12a}}$$

$$Q_a := \frac{\sqrt{b_{2a}}}{b_{1a}} = 0.056 \quad \omega_{0a} := \sqrt{b_{2a}} \quad f_{0a} := \frac{\omega_{0a}}{2 \cdot \pi} = 82.26 \cdot \text{kHz}$$

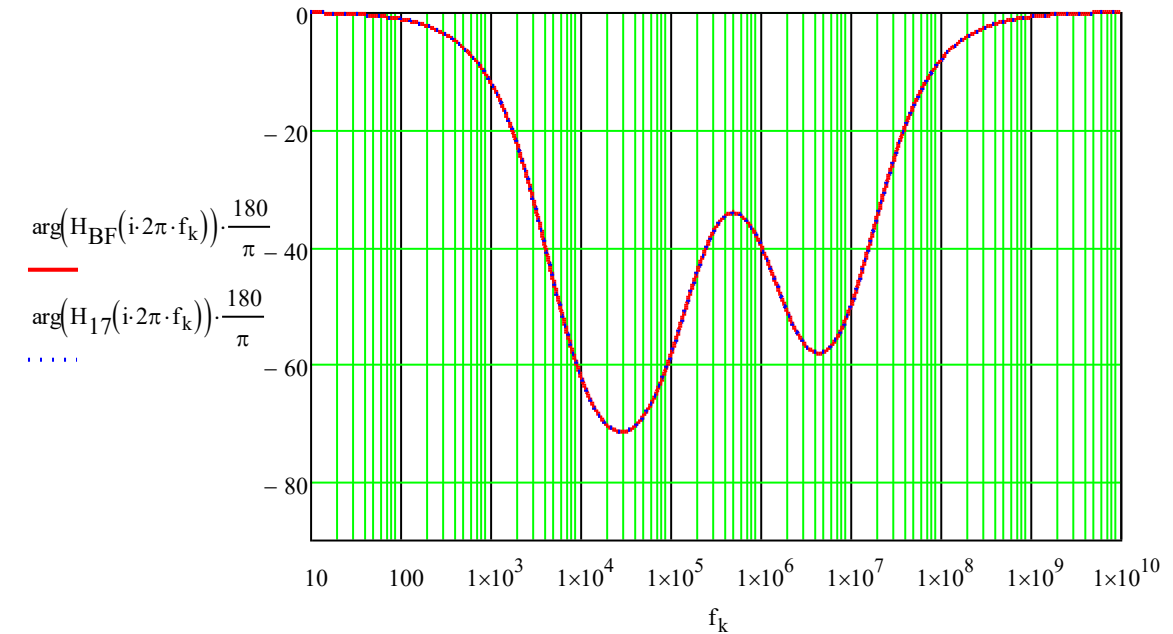
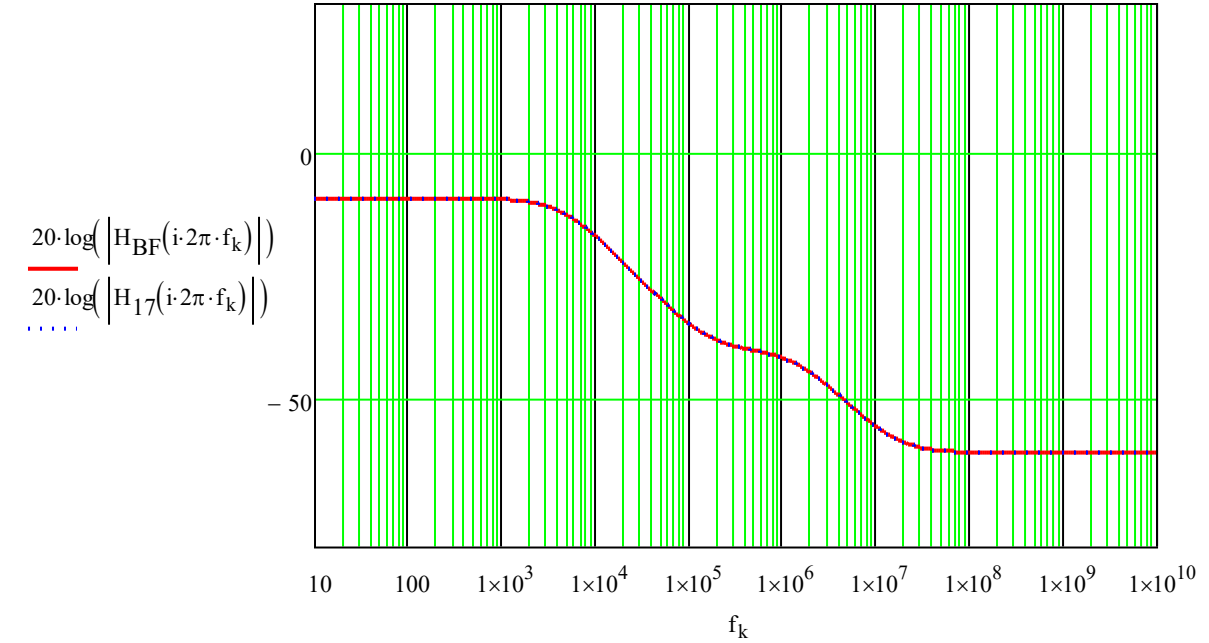
$$D_{2a}(s) := 1 + \frac{\omega_{0a}}{s \cdot Q_a} + \left( \frac{\omega_{0a}}{s} \right)^2$$

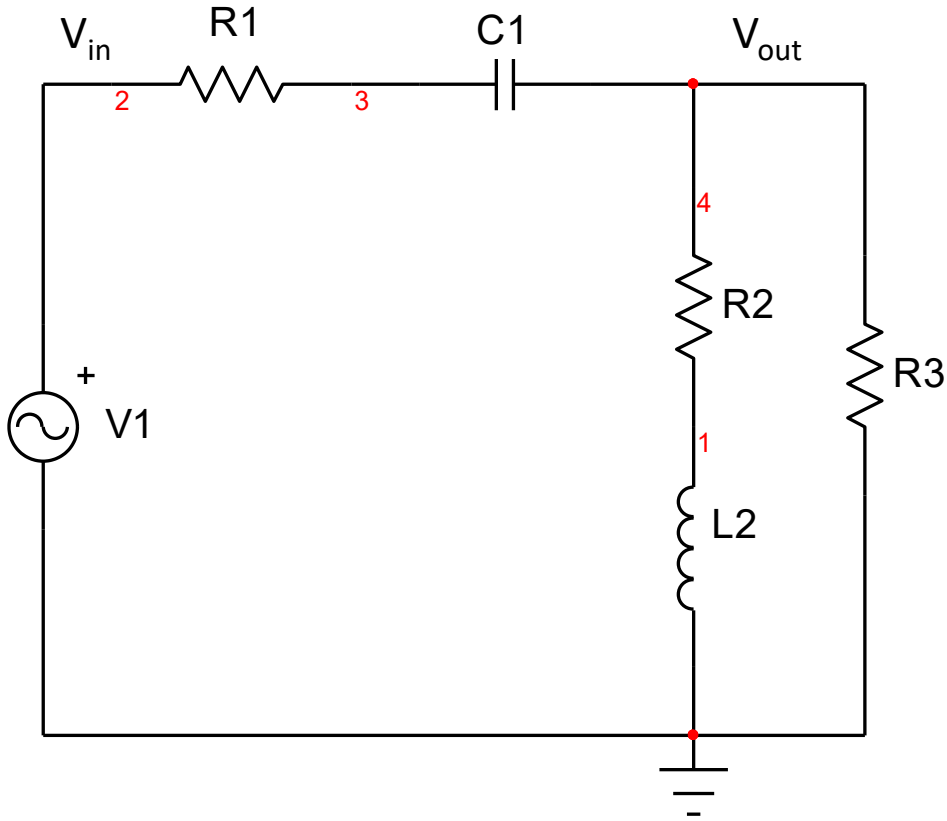
$$H_{16}(s) := H_{\text{inf}} \cdot \frac{\left( 1 + \frac{\omega_{z1}}{s} \right) \cdot \left( 1 + \frac{\omega_{z2}}{s} \right)}{1 + \frac{b_{1a}}{s} + \frac{b_{2a}}{s^2}}$$

$$H_{18}(s) := H_{\text{inf}} \cdot \frac{N_{1a}(s)}{D_{1a}(s)}$$

$$H_{15}(s) := H_{\text{inf}} \cdot \frac{\left( 1 + \frac{\omega_{z1}}{s} \right) \cdot \left( 1 + \frac{\omega_{z2}}{s} \right)}{1 + \left( \frac{1}{\tau_{1a}} + \frac{1}{\tau_{2a}} \right) \cdot \frac{1}{s} + \left( \frac{1}{\tau_{1a} \cdot \tau_{12a}} \right) \cdot \frac{1}{s^2}}$$

$$H_{17}(s) := H_{\text{inf}} \cdot \frac{\left( 1 + \frac{\omega_{z1}}{s} \right) \cdot \left( 1 + \frac{\omega_{z2}}{s} \right)}{1 + \frac{\omega_{0a}}{s \cdot Q_a} + \left( \frac{\omega_{0a}}{s} \right)^2}$$

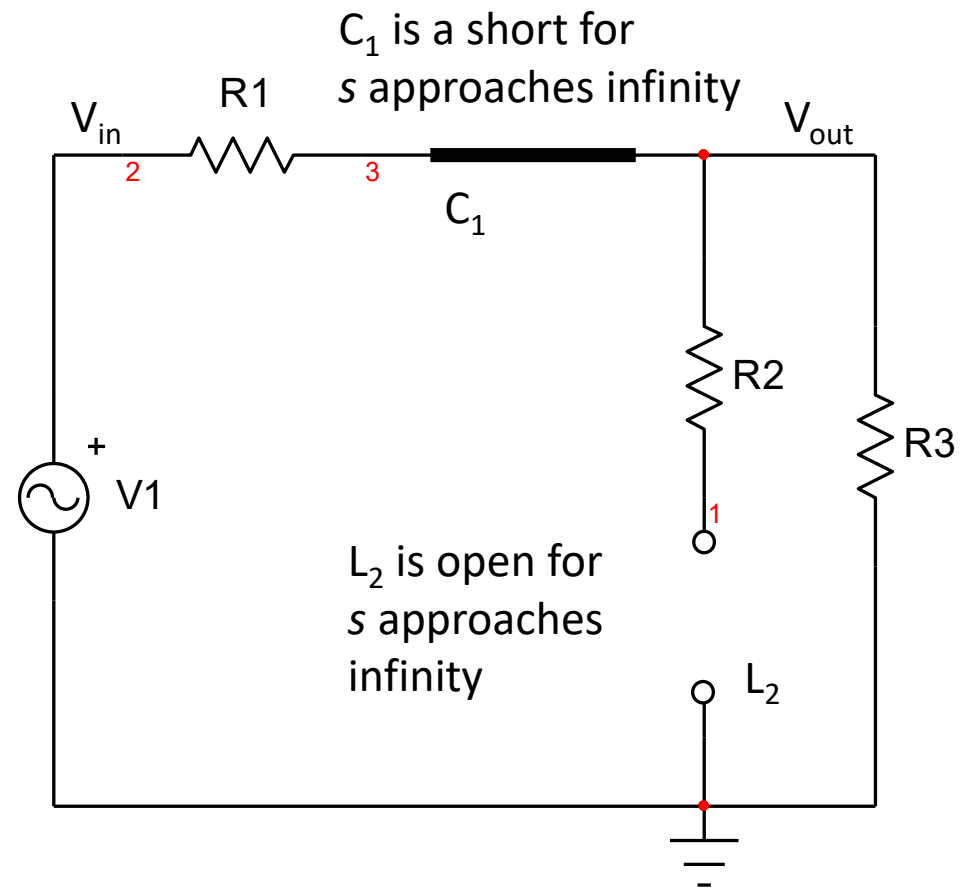




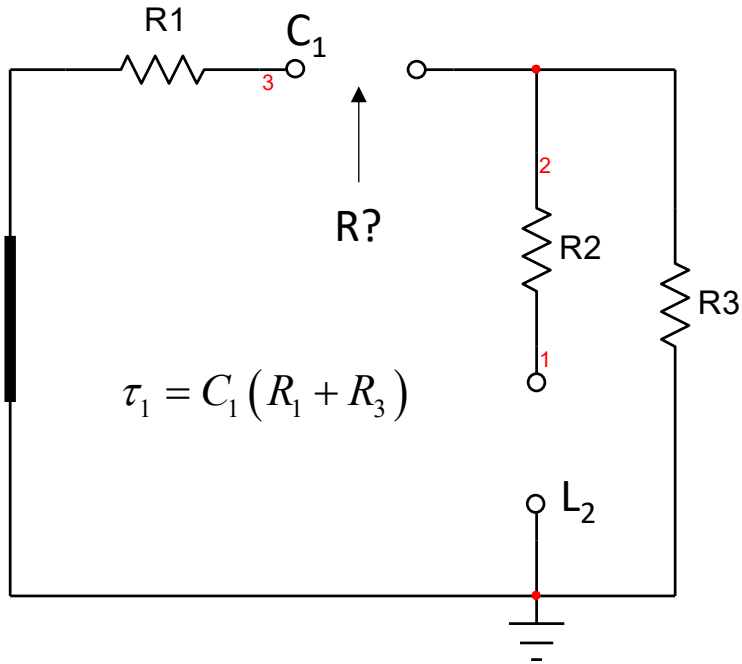
Reference state:  
 $C_1$  is a short.  
 Inductance  $L_2$  is open

In this circuit, we can see that the dc gain is equal to zero.  
 However, the gain when  $s$  approaches infinity exists:

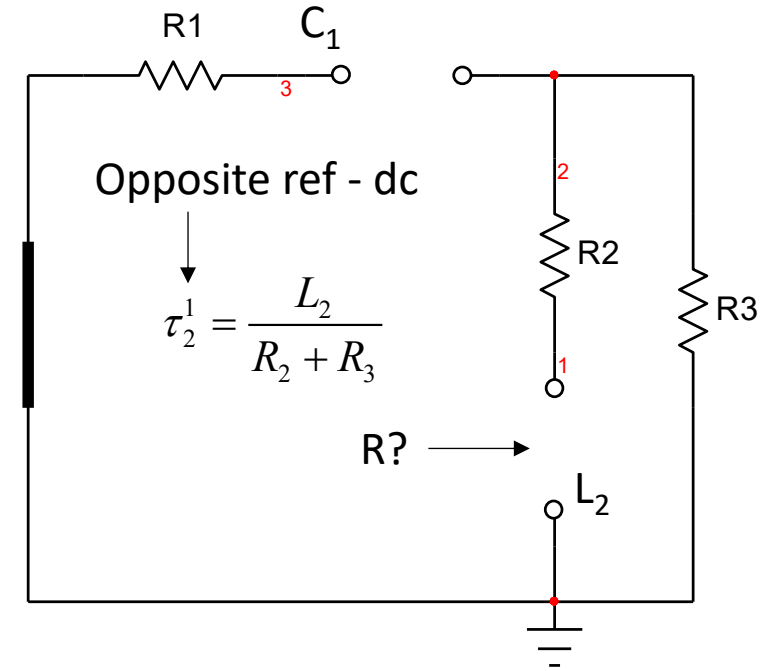
$$H_{INF} = \frac{R_3}{R_1 + R_3}$$



Determine tau1:

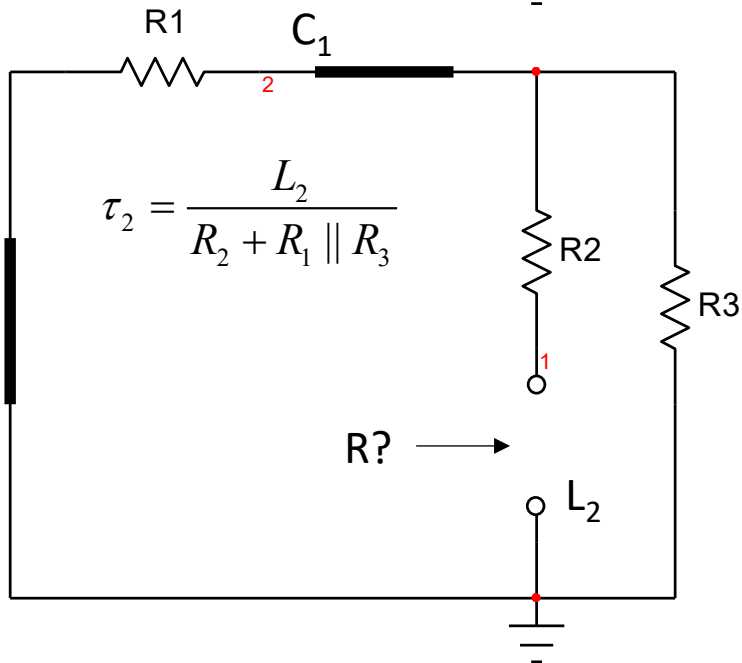


In this case,  $L_2$  is in its reference state which is an open circuit.



Determine tau2:

In this mode,  $C_1$  is in the opposite of its reference state (which is an open circuit).



In this case,  $C_1$  is in its reference state which is a short.

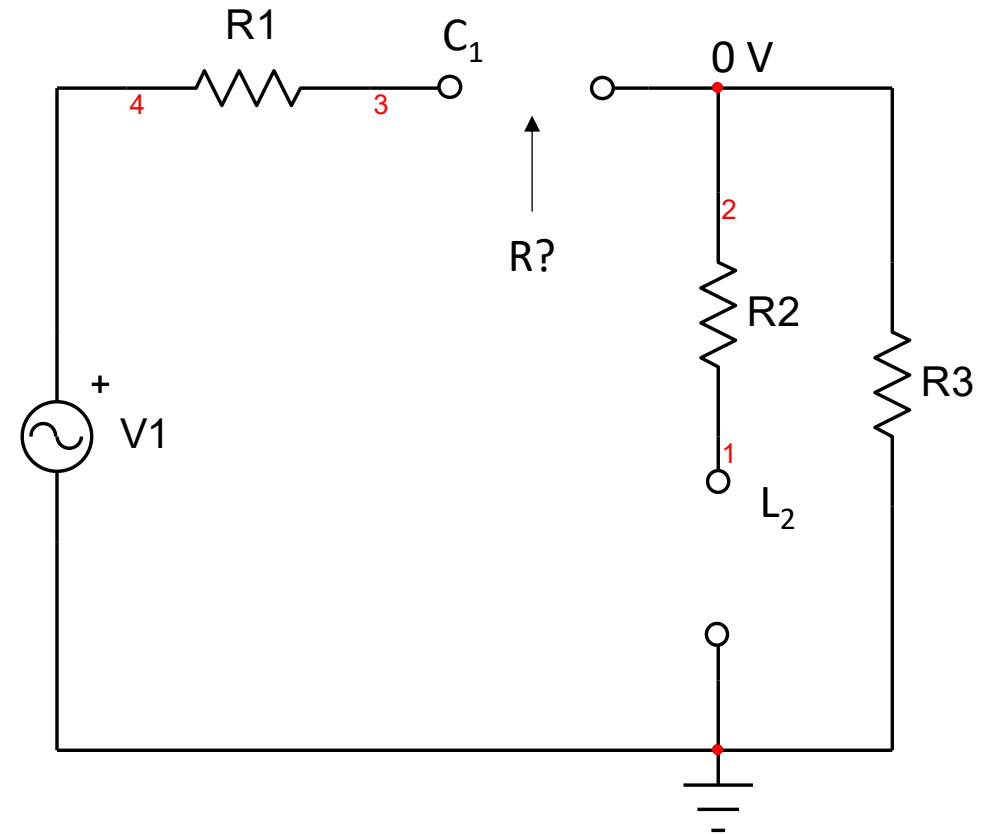
$$b_1 = \frac{1}{\tau_1} + \frac{1}{\tau_2} = \frac{1}{C_1(R_1 + R_3)} + \frac{R_2 + R_1 \parallel R_3}{L_2}$$

$$b_2 = \frac{1}{\tau_1 \tau_2^1} = \frac{1}{C_1(R_1 + R_3) \frac{L_2}{R_2 + R_3}} = \frac{1}{C_1 L_2} \frac{R_2 + R_3}{R_1 + R_3}$$

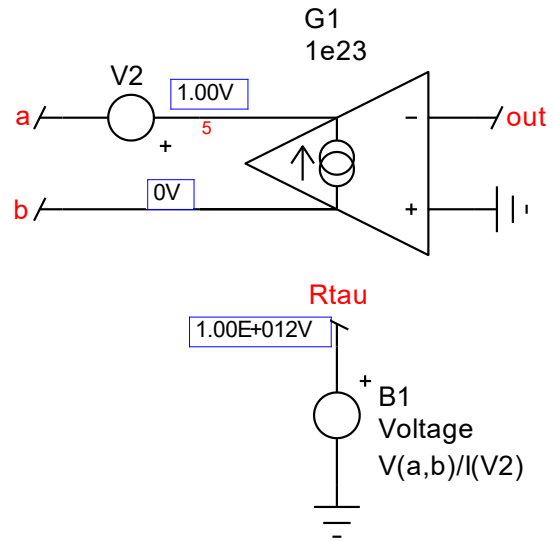
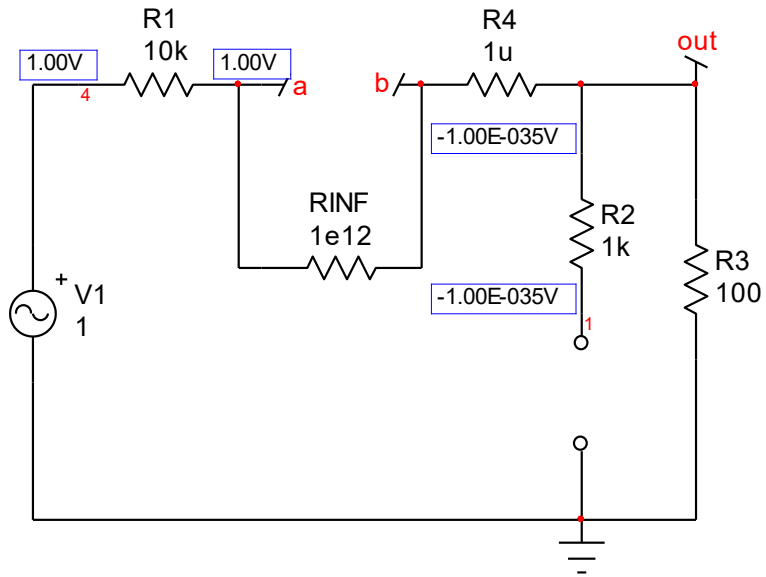


In this case, there is just one zero which is determined by inspection. The zero at the origin is merged with the inverted poles and zeroes. But we can perform the NDI to confirm this point.

Apply NDI to  $C_1$ , keeping  $V_{out}$  to 0 V ac.

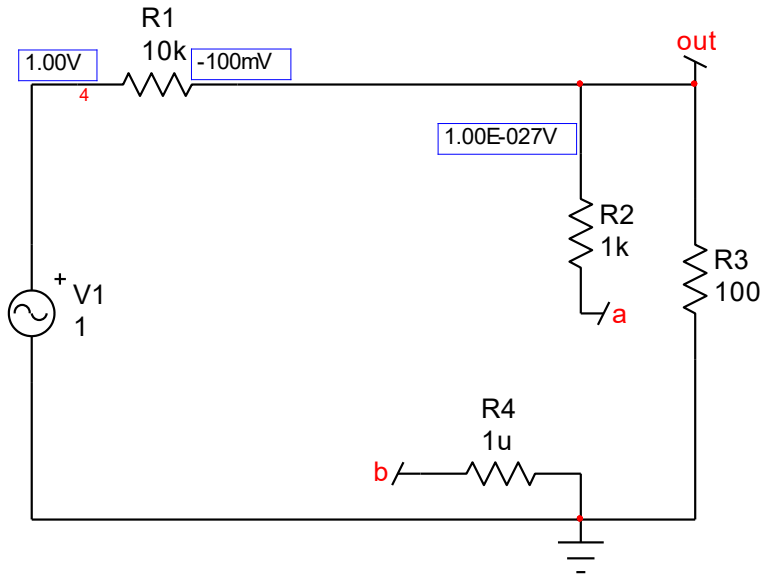


In this case, the resistance offered by the connecting terminals is infinite.

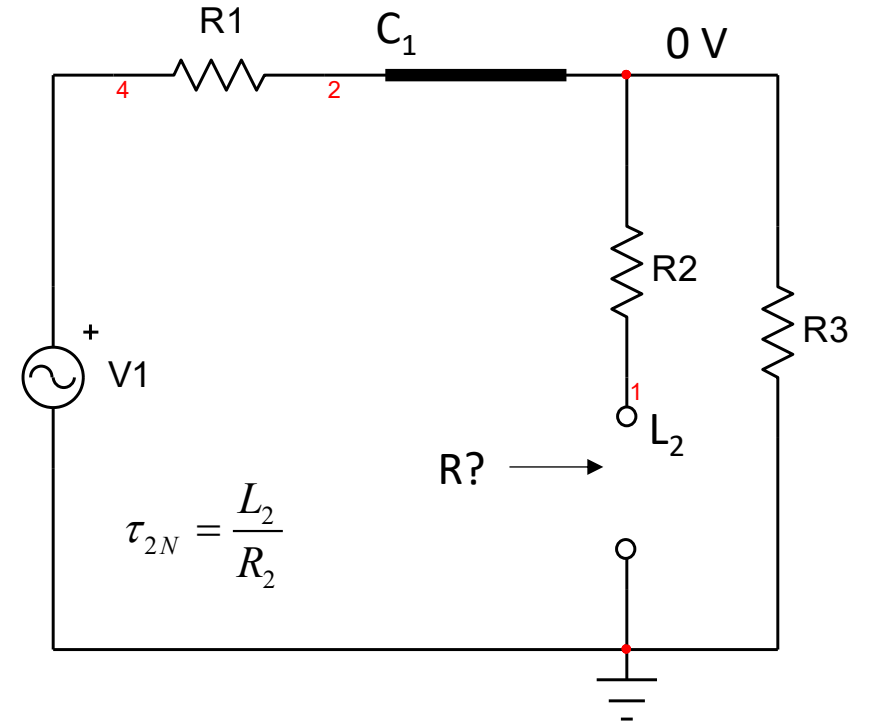
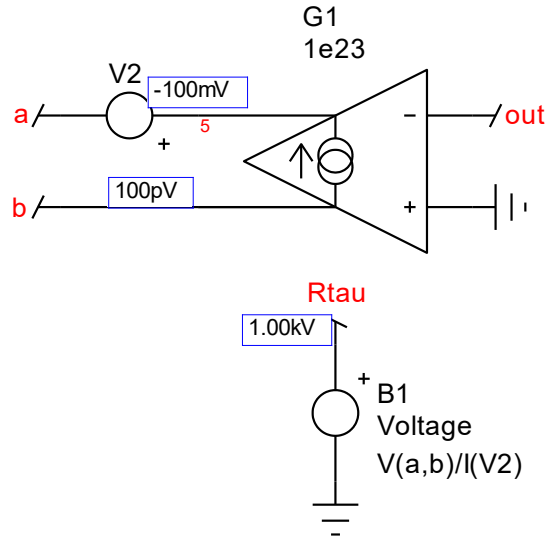


SPICE NDI confirms an infinite time constant that I make finite with resistance  $R_{INF}$ .

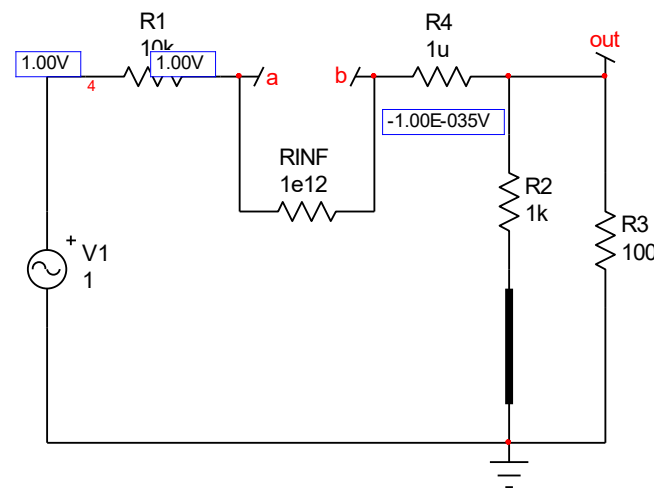
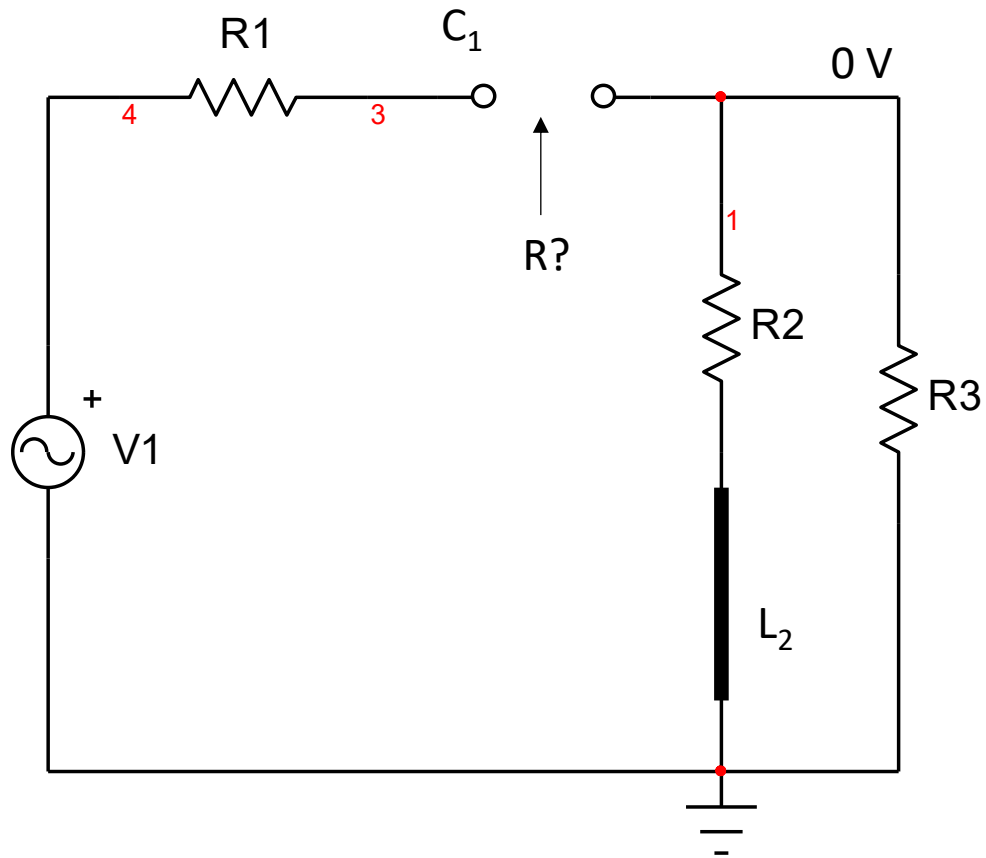
$$\tau_{1N} = C_1 R_{INF}$$



Confirmed by operating point

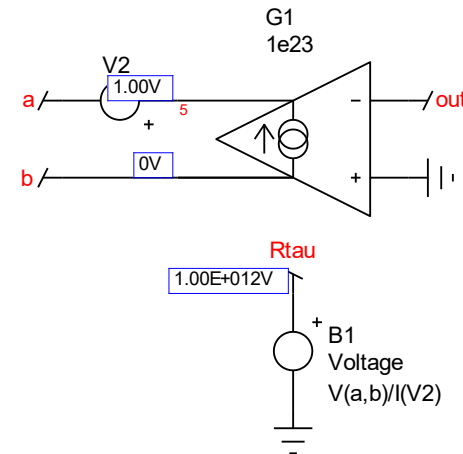


$$a_1 = \frac{1}{\tau_{1N}} + \frac{1}{\tau_{2N}} = \frac{1}{C_1 R_{INF}} + \frac{R_2}{L_2} = \frac{R_2}{L_2}$$



Opposite ref – short for  $L_2$

$$\tau_{1N}^2 = C_1 R_{INF}$$



$$a_2 = \frac{1}{\tau_{2N} \tau_{1N}^2} = \frac{1}{\frac{L_2}{R_2} C_1 R_{INF}} = 0$$



$$N(s) = 1 + \frac{a_1}{s} + \cancel{\frac{a_2}{s}} = 1 + \frac{R_2}{L_2 s} = 1 + \frac{\omega_z}{s}$$

$$b_1 = \frac{1}{\tau_1} + \frac{1}{\tau_2} = \frac{1}{C_1(R_1 + R_3)} + \frac{R_2 + R_1 \parallel R_3}{L_2}$$

$$b_2 = \frac{1}{\tau_1 \tau_2} = \frac{1}{C_1(R_1 + R_3) \frac{L_2}{R_2 + R_3}} = \frac{1}{C_1 L_2} \frac{R_2 + R_3}{R_1 + R_3}$$

$$D(s) = 1 + \frac{b_1}{s} + \frac{b_2}{s^2} = 1 + s \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) + \frac{1}{\tau_1 \tau_2 s^2} = 1 + \frac{\omega_0}{sQ} + \left( \frac{\omega_0}{s} \right)^2$$

$$Q = \frac{\sqrt{b_2}}{b_1} \quad \omega_0 = \sqrt{b_2}$$

$$N(s) = 1 + \frac{a_1}{s} + \frac{a_2}{s^2} = 1 + s \left( \frac{1}{\tau_{1N}} + \frac{1}{\tau_{2N}} \right) + \frac{1}{\tau_{1N} \tau_{2N} s^2}$$

$$H(s) = H_{INF} \frac{N(s)}{D(s)} = H_{INF} \frac{1 + \frac{s}{\omega_z}}{1 + \frac{\omega_0}{sQ} + \left( \frac{\omega_0}{s} \right)^2}$$

$$H_{22}(s) := \frac{R_3}{R_1 + R_3} \cdot \frac{1 + \frac{R_2}{s \cdot L_2}}{1 + \left[ \frac{1}{C_1 \cdot (R_1 + R_3)} + \frac{1}{L_2} \right] \cdot \frac{1}{s} + \left[ \frac{R_2 + R_3}{C_1 \cdot (R_1 + R_3) \cdot L_2} \right] \cdot \frac{1}{s^2}}$$

$$\|(x,y) := \frac{x \cdot y}{x + y} \quad R_{\text{INF}} := 10^{23} \Omega$$

$$R_1 := 10 \text{ k}\Omega \quad R_2 := 1 \text{ k}\Omega \quad R_3 := 22 \text{ k}\Omega \quad C_1 := 0.1 \mu\text{F} \quad L_2 := 10 \mu\text{H}$$

$$\tau_1 := C_1 \cdot (R_1 + R_3) = 3.2 \text{ ms}$$

$$H_{\text{INF}} := \frac{R_3}{R_1 + R_3} \quad 20 \cdot \log(H_{\text{INF}}) = -3.255$$

$$\tau_2 := \frac{L_2}{R_2 + R_1 \parallel R_3} = 1.27 \times 10^{-3} \cdot \mu\text{s}$$

$$\tau_{12} := \frac{L_2}{R_2 + R_3} = 4.348 \times 10^{-4} \cdot \mu\text{s}$$

$$\omega_z := \frac{R_2}{L_2}$$

$$\tau_{1N} := C_1 \cdot R_{\text{INF}} \quad \tau_{2N} := \frac{L_2}{R_2} \quad \tau_{21N} := C_1 \cdot R_{\text{INF}}$$

$$D_1(s) := 1 + \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) \cdot \frac{1}{s} + \left( \frac{1}{\tau_1 \cdot \tau_{12}} \right) \cdot \frac{1}{s^2}$$

$$N_1(s) := 1 + \left( \frac{1}{\tau_{1N}} + \frac{1}{\tau_{2N}} \right) \cdot \frac{1}{s} + \left( \frac{1}{\tau_{2N} \cdot \tau_{21N}} \right) \cdot \frac{1}{s^2}$$

$$b_1 := \frac{1}{\tau_1} + \frac{1}{\tau_2} \quad b_2 := \frac{1}{\tau_1 \cdot \tau_{12}} = 7.188 \times 10^{11} \frac{1}{s^2}$$

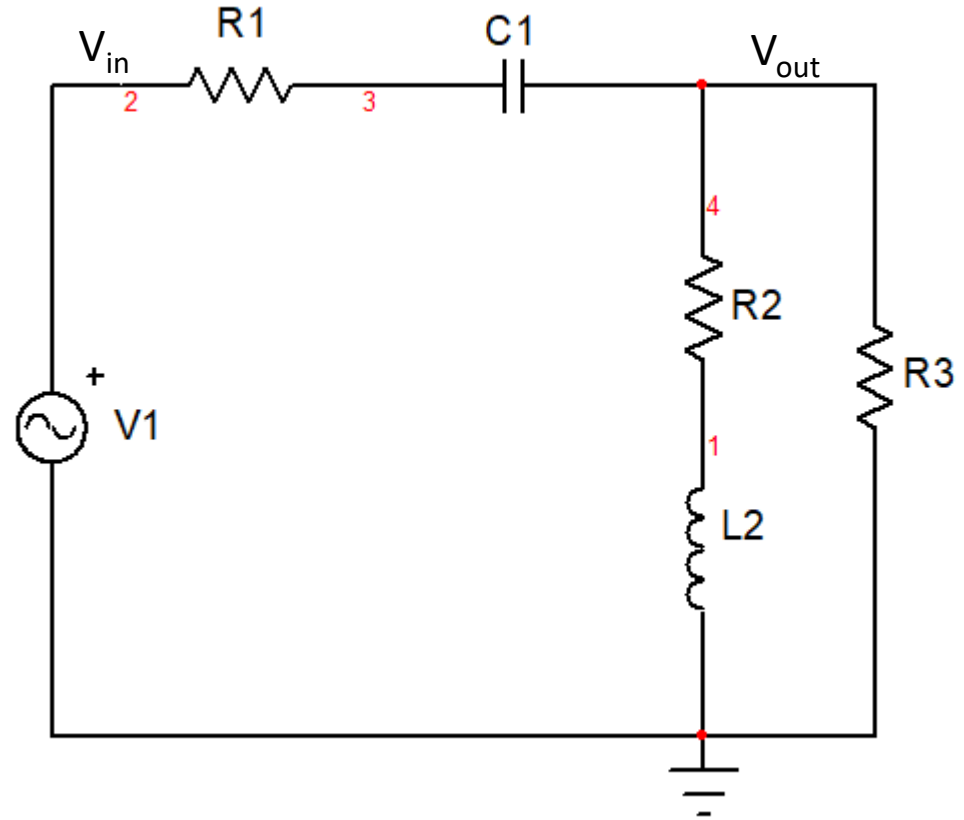
$$a_1 := \frac{1}{\tau_{1N}} + \frac{1}{\tau_{2N}} \quad a_2 := \frac{1}{\tau_{2N} \cdot \tau_{21N}}$$

$$N_3(s) := 1 + \frac{a_1}{s} + \frac{a_2}{s^2} \quad D_3(s) := 1 + \frac{b_1}{s} + \frac{b_2}{s^2}$$

$$Q := \frac{\sqrt{b_2}}{b_1} = 1.077 \times 10^{-3} \quad \omega_0 := \sqrt{b_2} \quad f_0 := \frac{\omega_0}{2 \cdot \pi} = 134.93 \text{ kHz}$$

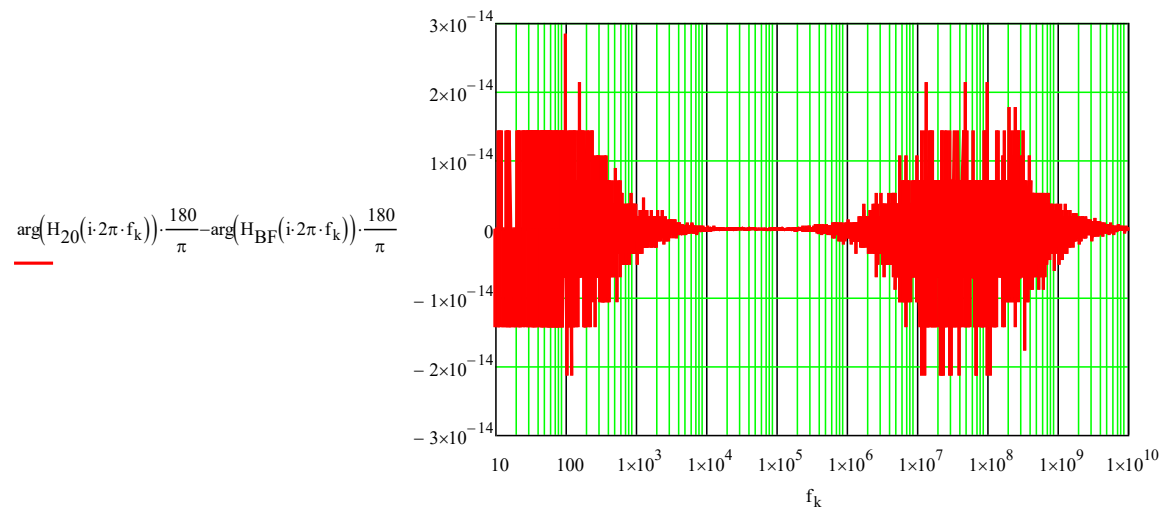
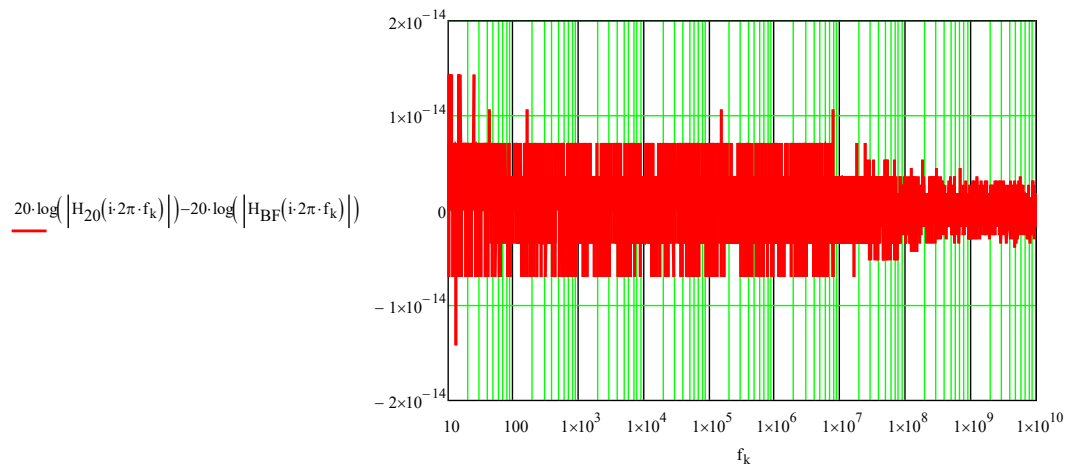
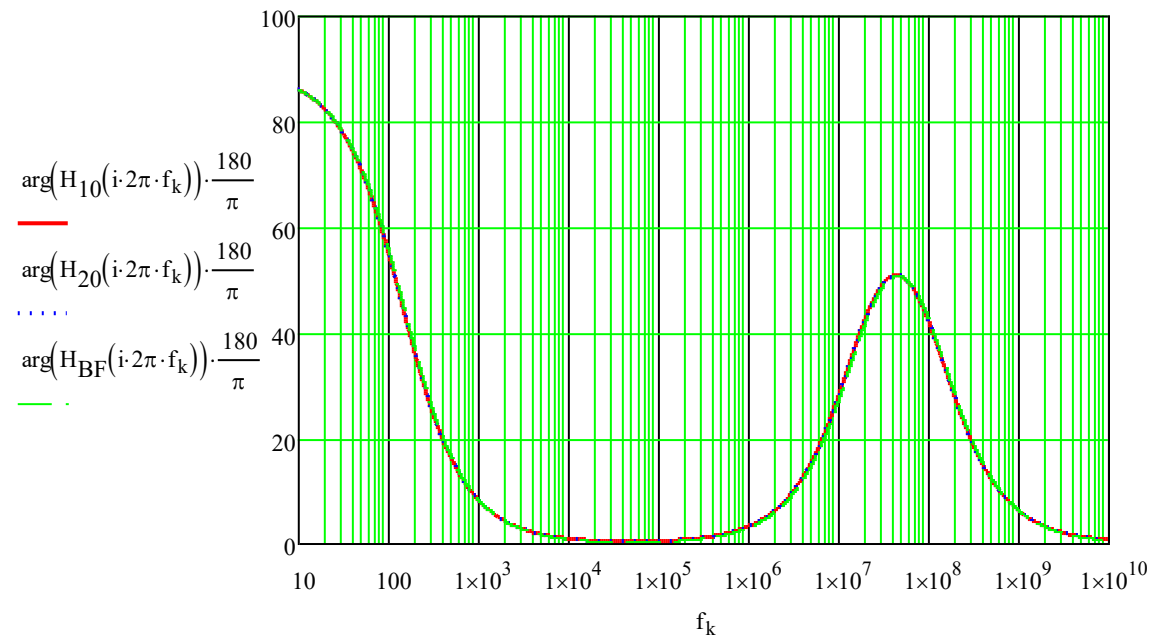
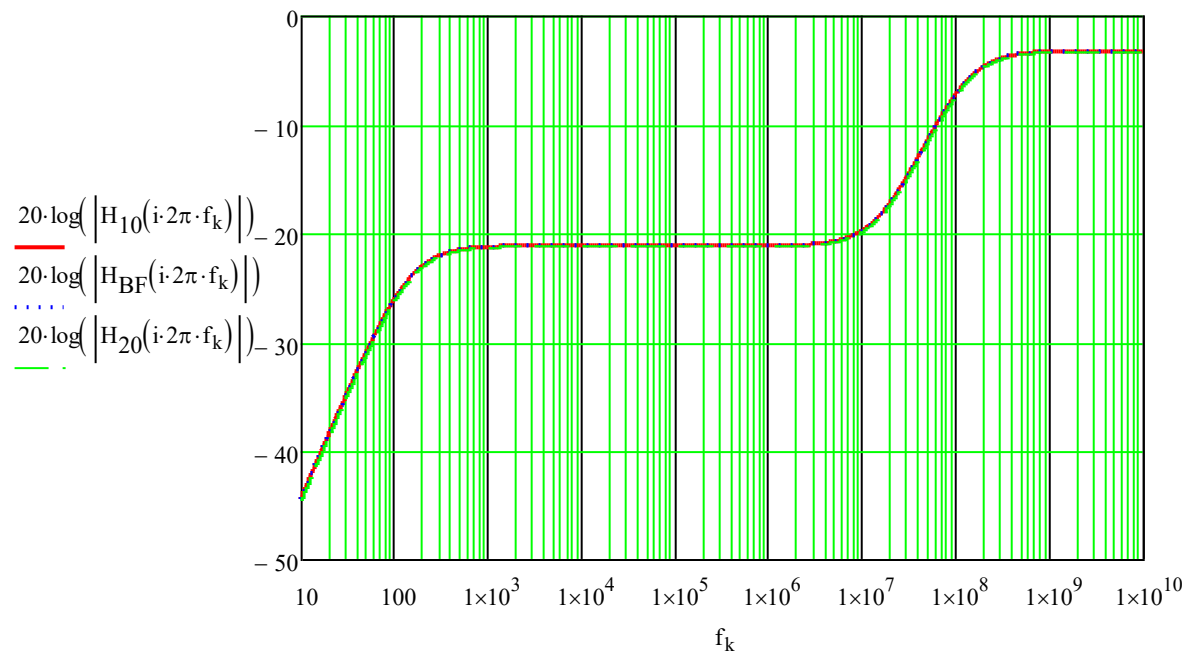
$$D_4(s) := 1 + \frac{\omega_0}{s \cdot Q} + \left( \frac{\omega_0}{s} \right)^2 \quad H_{10}(s) := H_{\text{INF}} \cdot \frac{N_1(s)}{D_1(s)}$$

$$H_{20}(s) := H_{\text{INF}} \cdot \frac{1 + \frac{\omega_z}{s}}{1 + \frac{\omega_0}{s \cdot Q} + \left( \frac{\omega_0}{s} \right)^2}$$



Raw brute-force TF:

$$H_{\text{BF}}(s) := \frac{[R_3 \parallel (R_2 + s \cdot L_2)]}{R_1 + \frac{1}{s \cdot C_1} + [R_3 \parallel (R_2 + s \cdot L_2)]}$$



In my 2016 [book](#), I have illustrated how a 2<sup>nd</sup>-order circuit transfer function could be expressed in four different transfer functions, depending on the various states adopted for the reference state  $H_{\text{ref}}$ :

$Z_1$  and  $Z_2$  are both considered for  $s = 0$

$Z_1$  and  $Z_2$  are both considered for  $s \rightarrow \text{infinity}$

$Z_1$  is considered for  $s = 0$  and  $Z_2$  for  $s \rightarrow \text{infinity}$

$Z_1$  is considered for  $s \rightarrow \text{infinity}$  and  $Z_2$  for  $s = 0$

For practical reasons, I realized that the first case (all impedances considered for  $s = 0$ ) was the most intuitive and practical for engineers as it represents a natural state in the lab for a dc-biased circuit or when determining an operating bias point in simulation (all caps open and inductors shorted).

In some cases – especially when one or several zeroes at the origin exist (no dc path) – the leading term determined at  $s = 0$  is zero. It is therefore possible to factor the original second-order transfer function and have a leading term determined for  $s$  approaches infinity instead, using inverted poles and zeroes in this case. These are the examples I have documented in this PPT.

After all, the goal of the FACTs is to swiftly determine transfer functions by breaking down a complicated circuit in smaller and simpler schematic diagrams. To that respect, considering  $s$  approaches infinity for all the elements, is a possible option. But if the process remains reasonably simple for a 2<sup>nd</sup>-order system (you juggle with two energy-storing elements), it seriously complicates with more elements, especially if you mix a combination of reference states for the energy-storing elements.